

RIESZ s -EQUILIBRIUM MEASURES ON d -RECTIFIABLE SETS AS s APPROACHES d

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ABSTRACT. Let A be a compact set in \mathbb{R}^p of Hausdorff dimension d . For $s \in (0, d)$, the Riesz s -equilibrium measure μ^s is the unique Borel probability measure with support in A that minimizes

$$I_s(\mu) := \iint \frac{1}{|x - y|^s} d\mu(y) d\mu(x)$$

over all such probability measures. If A is strongly (\mathcal{H}^d, d) -rectifiable, then μ^s converges in the weak-star topology to normalized d -dimensional Hausdorff measure restricted to A as s approaches d from below.

Riesz potential, equilibrium measure, d -rectifiable

1. INTRODUCTION

Let A be a compact subset of \mathbb{R}^p with positive and finite d -dimensional Hausdorff measure $\mathcal{H}^d(A)$. Let $\mathcal{M}(A)$ denote the set of Radon measures with support in A , and $\mathcal{M}_1(A) \subset \mathcal{M}(A)$ the Borel probability measures with support in A . The Riesz s -energy of a measure $\mu \in \mathcal{M}(A)$ is defined by

$$I_s(\mu) := \iint \frac{1}{|x - y|^s} d\mu(y) d\mu(x).$$

If $s \in (0, d)$, then there is a unique measure $\mu^s = \mu^{s,A} \in \mathcal{M}_1(A)$ called the (s) -equilibrium measure on A such that $I_s(\mu^s) < I_s(\nu)$ for any measure $\nu \in \mathcal{M}_1(A) \setminus \{\mu^s\}$, while, for $s \geq d$, $I_s(\nu) = \infty$ for any non-trivial measure $\nu \in \mathcal{M}(A)$ (cf. [8, 9]). The uniqueness of the equilibrium measure arises from the positivity of the Riesz kernel (cf. [4, 8]). For example, in the case that A is the interval $[-1, 1]$ and $s \in (0, 1)$, it is well-known (cf. [6]) that $d\mu^s(x) = c_s(1 - x^2)^{\frac{s-1}{2}} dx$ where c_s is chosen so that μ^s is a probability measure.

In this paper, we investigate the behavior of μ^s as s approaches d from below. For $A = [-1, 1]$, we see directly from the above expression that μ^s converges in the weak-star sense as $s \uparrow 1$ to normalized Lebesgue measure restricted to A . It is natural to ask how general is this phenomena. We are further motivated by recent results concerning the following related discrete minimal energy problem. For a configuration of $N \geq 2$ points $\omega_N := \{x_1, \dots, x_N\}$ and $s > 0$, the Riesz s -energy of ω_N is defined by

$$E_s(\omega_N) := \sum_{\substack{i,j=1 \\ i \neq j}}^N \frac{1}{|x_i - x_j|^s}.$$

The compactness of A and the lower semicontinuity of the Riesz kernel imply that there is a (not necessarily unique) configuration $\omega_N^s \subset A$ that minimizes E_s over all N -point configurations on A . When $s < d$ the above continuous and discrete problems are related by the following two results (cf. [8]). First, $E_s(\omega_N^s)/N^2 \rightarrow I_s(\mu^s)$ as $N \rightarrow \infty$. Second, the sequence of configurations $\{\omega_N^s\}_{N=1}^\infty$ has asymptotic distribution μ^s , that is, the sequence of discrete measures

$$\mu^{s,N} := \frac{1}{N} \sum_{x \in \omega_N^s} \delta_x$$

(where δ_x denotes the unit atomic measure at x) converges to μ^s in the weak-star topology on $\mathcal{M}(A)$ as $N \rightarrow \infty$. We use a starred arrow to denote weak-star convergence, that is, for $s \in (0, d)$ we have

$$(1) \quad \mu^{s,N} \xrightarrow{*} \mu^s \quad \text{as } N \rightarrow \infty.$$

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In the case $s \geq d$, the discrete minimal energy problem is well-posed even though the continuous problem is not. Recently, asymptotic results for the discrete minimal energy problem were obtained in [5] and [2] for this range of s and the case that A is a d -rectifiable set where A is d -rectifiable (cf. [3, §3.2.14]) if it is the Lipschitz image of a bounded set in \mathbb{R}^d . In this case,

$$(2) \quad \mu^{s,N} \xrightarrow{*} \mathcal{H}_A^d / \mathcal{H}^d(A) \quad \text{as } N \rightarrow \infty.$$

(Here and in the rest of the paper we use the notation μ_E to denote the restriction of a measure μ to a μ -measurable set E . e.g. $\mathcal{H}_A^d = \mathcal{H}^d(\cdot \cap A)$.) For technical reasons, the results in [5] and [2] for the case $s = d$ further require that A be a subset of a d -dimensional C^1 manifold, although it is conjectured that this hypothesis is unnecessary.

The limits (1) and (2) suggest that $\mu^s \xrightarrow{*} \mathcal{H}_A^d / \mathcal{H}^d(A)$ as $s \uparrow d$ whenever A is d -rectifiable. If A is strongly (\mathcal{H}^d, d) -rectifiable (see Definition 1.1 below), we show that this is indeed the case. A primary tool in our work is the following normalized d -energy of a measure

$$\tilde{I}_d(\mu) := \lim_{s \uparrow d} (d - s) I_s(\mu),$$

which we show is well-defined for every measure $\mu \in \mathcal{M}(A)$ and is uniquely minimized over $\mathcal{M}_1(A)$ by the measure $\lambda^d := \mathcal{H}_A^d / \mathcal{H}^d(A)$.

A map $f : A \rightarrow \mathbb{R}^p$ is *Lipschitz* if there is a constant L such that, for any $x, y \in A$,

$$|f(x) - f(y)| < L|x - y|,$$

and is *bi-Lipschitz* if there is a constant L such that for any $x, y \in A$,

$$\frac{1}{L}|x - y| < |f(x) - f(y)| < L|x - y|.$$

A set $A \subset \mathbb{R}^p$ is (\mathcal{H}^d, d) -rectifiable (cf. [3, §3.2.14]) if $\mathcal{H}^d(A) < \infty$ and there exists a countable collection E_1, E_2, \dots of d -rectifiable sets that cover \mathcal{H}^d -almost all of A . That is, there exists a countable collection of bounded subsets of \mathbb{R}^d K_1, K_2, \dots and a corresponding collection of Lipschitz maps, $\varphi_1 : K_1 \rightarrow \mathbb{R}^p, \varphi_2 : K_2 \rightarrow \mathbb{R}^p, \dots$ such that

$$\mathcal{H}^d\left(A \setminus \bigcup_{i=1}^{\infty} \varphi_i(K_i)\right) = 0.$$

Moreover, it is a result of Federer [3, §3.2.18]) that if A is (\mathcal{H}^d, d) -rectifiable then for every $\varepsilon > 0$, the Lipschitz maps and the bounded sets may be chosen such that each φ_i is bi-Lipschitz with constant less than $1 + \varepsilon$, each K_i is compact and the sets $\varphi_1(K_1), \varphi_2(K_2), \dots$ are pairwise disjoint. For such a choice of the φ_i and K_i there is an $N = N(\varepsilon)$ such that

$$\mathcal{H}^d\left(A \setminus \bigcup_{i=1}^N \varphi_i(K_i)\right) < \varepsilon.$$

The following definition of strong (\mathcal{H}^d, d) -rectifiability strengthens this condition in that for each $\varepsilon > 0$ there must be a finite collection of the mappings as above such that the portion of A not covered by the union is of strictly lower dimension.

Definition 1.1. We say that a set $A \subset \mathbb{R}^p$ is strongly (\mathcal{H}^d, d) -rectifiable if, for every $\varepsilon > 0$, there is a finite collection of compact subsets of \mathbb{R}^d K_1, \dots, K_N and a corresponding set of bi-Lipschitz maps $\varphi_1 : K_1 \rightarrow \mathbb{R}^p, \dots, \varphi_N : K_N \rightarrow \mathbb{R}^p$ such that

1. The bi-Lipschitz constant of each map is less than $1 + \varepsilon$.
2. $\mathcal{H}^d(\varphi_i(K_i) \cap \varphi_j(K_j)) = 0$ for all $i \neq j$.
3. $\dim\left(A \setminus \bigcup_{i=1}^N \varphi_i(K_i)\right) < d$.

Note that compact subsets of d -dimensional C^1 manifolds are strongly (\mathcal{H}^d, d) -rectifiable and any strongly (\mathcal{H}^d, d) -rectifiable set is (\mathcal{H}^d, d) -rectifiable.

The Riesz s -potential of a measure $\mu \in \mathcal{M}(A)$ at a point $x \in \mathbb{R}^p$ is given by

$$U_s^\mu(x) := \int \frac{1}{|x - y|^s} d\mu(y).$$

If $\mu \in \mathcal{M}_1(A)$, then $\lim_{s \uparrow d} U_s^\mu(x) = \infty$ for all x in some set of positive μ -measure and hence, as mentioned previously, $\lim_{s \uparrow d} I_s(\mu) = \infty$ (cf. [9]). For $x \in \mathbb{R}^p$, we define (when it exists) the following normalized d -potential

$$(3) \quad \tilde{U}_d^\mu(x) := \lim_{s \uparrow d} (d - s) U_s^\mu(x).$$

In certain cases, $\tilde{U}_d^\mu(x)$ behaves like an average density of μ at x . In particular, it is shown by Hinz in [7] that, if there is a constant $C = C(x)$ such that $\mu(B(x, r)) < Cr^d$ for all $r > 0$, then $\tilde{U}_d^\mu(x)$ equals d times the order-two density defined by Bedford and Fisher in [1]

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|\ln \varepsilon|} \int_\varepsilon^1 \frac{\mu(B(x, r))}{r^d} \frac{1}{r} dr$$

at any point x where this limit exists. This in turn equals the usual density

$$D_\mu(x) := \lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^d}$$

at any point x where $D_\mu(x)$ exists. (Here $B(x, r)$ denotes the closed unit ball centered at x of radius r . The corresponding open ball is denoted $B(x, r)^0$.) Note that the order-two density exists for many measures for which the density does not (cf. [12] and references therein).

Finally, we remark that in [10] M. Putinar considered a different normalized Riesz d -potential in his work on solving inverse moment problems.

1.1. Main Results. Our first theorem asserts that the normalized d -energy \tilde{I}_d is well defined and gives rise to a minimization problem with a unique solution. Note that we choose a normalization of \mathcal{H}^d so that the \mathcal{H}^d -measure of $B(0, 1) \subset \mathbb{R}^d$ is 2^d .

Theorem 1.2. *Let $A \subset \mathbb{R}^p$ be compact and strongly (\mathcal{H}^d, d) -rectifiable such that $\mathcal{H}^d(A) > 0$. Let $\lambda^d := \mathcal{H}_A^d / \mathcal{H}^d(A)$. Then*

1. $\tilde{I}_d(\mu)$ exists as an extended real number for every measure $\mu \in \mathcal{M}(A)$ and

$$\tilde{I}_d(\mu) = \begin{cases} 2^d d \int \left(\frac{d\mu}{d\mathcal{H}_A^d} \right)^2 d\mathcal{H}_A^d & \text{if } \mu \ll \mathcal{H}_A^d, \\ \infty & \text{otherwise.} \end{cases}$$

2. If, for some measure $\mu \in \mathcal{M}(A)$ $\tilde{I}_d(\mu) < \infty$, then \tilde{U}_d^μ exists and is finite μ -a.e. and

$$\tilde{I}_d(\mu) = \int \tilde{U}_d^\mu d\mu.$$

3. $\tilde{I}_d(\lambda^d) < \tilde{I}_d(\nu)$ for every measure $\nu \in \mathcal{M}_1(A) \setminus \{\lambda^d\}$.

The second theorem asserts the weak-star convergence of the s -equilibrium measures to normalized Hausdorff measure as s approaches d from below. The essential idea behind the proof is that any weak-star limit point of the s -equilibrium measures, as s approaches d , has normalized d -energy less than or equal to that of λ^d .

Theorem 1.3. *Let $A \subset \mathbb{R}^p$ be compact and strongly (\mathcal{H}^d, d) -rectifiable such that $\mathcal{H}^d(A) > 0$. Let $\lambda^d := \mathcal{H}_A^d / \mathcal{H}^d(A)$. Then $\mu^s \xrightarrow{*} \lambda^d$ as $s \uparrow d$.*

The remainder of this paper is organized as follows. In Section 2 we prove several lemmas leading to a proof of Theorem 1.2. In Section 3 we show that μ^s converges to λ^d first, for the simpler case where A is a d -dimensional compact subset of \mathbb{R}^d . Then, by gluing together near isometries of compact subsets of \mathbb{R}^d , the theorem is proven for the more general case where A is a strongly (\mathcal{H}^d, d) -rectifiable subset of \mathbb{R}^p .

2. THE EXISTENCE OF A UNIQUE MINIMIZER OF \tilde{I}_d

In this paper, the Fourier transform of a finite Borel measure μ supported on \mathbb{R}^d is defined by

$$\mathbb{R}^d \ni \xi \rightarrow \hat{\mu}(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} d\mu(x).$$

For a compactly supported Radon measure μ on \mathbb{R}^d and $s \in (0, d)$ the Riesz s -energy of μ may be expressed as (cf. [8, 9, 11])

$$I_s(\mu) = c(s, d) \int_{\mathbb{R}^d} |\xi|^{s-d} |\hat{\mu}(\xi)|^2 d\mathcal{L}^d(\xi),$$

where \mathcal{L}^d denotes Lebesgue measure on \mathbb{R}^d and the constant $c(s, d)$ is given by

$$c(s, d) = \pi^{s-\frac{d}{2}} \frac{\Gamma(\frac{d-s}{2})}{\Gamma(\frac{s}{2})}.$$

Observe that (cf. [8, ch. 1])

$$(4) \quad \lim_{s \uparrow d} (d-s)c(s, d) = \omega_d,$$

where ω_d is the surface area of the $d-1$ sphere in \mathbb{R}^d .

Lemma 2.1. *Let $K \subset \mathbb{R}^d$ be compact. For a measure $\mu \in \mathcal{M}(K)$ we have*

$$\tilde{I}_d(\mu) = \omega_d \|\hat{\mu}\|_{2, \mathcal{L}^d}^2.$$

Further, if $\tilde{I}_d(\mu) < \infty$, then $\mu \ll \mathcal{L}^d$.

Proof. For any measure $\mu \in \mathcal{M}(K)$ the Riesz s -energy can be expressed as

$$I_s(\mu) = c(s, d) \int_{|\xi| \leq 1} |\xi|^{s-d} |\hat{\mu}(\xi)|^2 d\mathcal{L}^d(\xi) + c(s, d) \int_{|\xi| > 1} |\xi|^{s-d} |\hat{\mu}(\xi)|^2 d\mathcal{L}^d(\xi).$$

By dominated convergence

$$\lim_{s \uparrow d} \int_{|\xi| \leq 1} |\xi|^{s-d} |\hat{\mu}(\xi)|^2 d\mathcal{L}^d(\xi) = \int_{|\xi| \leq 1} |\hat{\mu}(\xi)|^2 d\mathcal{L}^d(\xi),$$

and by monotone convergence

$$\lim_{s \uparrow d} \int_{|\xi| > 1} |\xi|^{s-d} |\hat{\mu}(\xi)|^2 d\mathcal{L}^d(\xi) = \int_{|\xi| > 1} |\hat{\mu}(\xi)|^2 d\mathcal{L}^d(\xi).$$

From (4) the first statement is proven.

An established result (cf. [11]) is that, if $\hat{\mu} \in L^2(\mathcal{L}^d)$, then $\mu \ll \mathcal{L}^d$ and $d\mu/d\mathcal{L}^d \in L^2(\mathcal{L}^d)$. \square

Definition 2.2. (cf. [9, ch. 1]) *Let μ be a compactly supported Radon measure on \mathbb{R}^p and let $\varphi : \text{supp}\{\mu\} \rightarrow \mathbb{R}^{p'}$ be continuous. The image measure associated with μ and φ is the set-valued function $\varphi_{\#}\mu$ defined by*

$$\varphi_{\#}\mu(E) := \mu(\varphi^{-1}(E)).$$

The following are straightforward consequences of the above definition.

1. $\varphi_{\#}\mu$, as defined above, is a compactly supported Radon measure on $\mathbb{R}^{p'}$.
2. For a non-negative $\varphi_{\#}\mu$ -measurable function f

$$\int f d\varphi_{\#}\mu = \int f(\varphi) d\mu.$$

For $A \subset \mathbb{R}^p$, a bi-Lipschitz map $\varphi : A \rightarrow \mathbb{R}^{p'}$ with constant L , and a measure $\mu \in \mathcal{M}(A)$ it follows that

$$(5) \quad \frac{1}{L^s} I_s(\varphi_{\#}\mu) \leq I_s(\mu) \leq L^s I_s(\varphi_{\#}\mu),$$

and

$$(6) \quad \frac{1}{L^d} \mathcal{H}^d(\varphi(A)) \leq \mathcal{H}^d(A) \leq L^d \mathcal{H}^d(\varphi(A)).$$

Note (6) implies $\mu \perp \mathcal{H}^d$ if and only if $\varphi_{\#}\mu \perp \mathcal{H}^d$.

Lemma 2.3. *Let $A \subset \mathbb{R}^p$ be strongly (\mathcal{H}^d, d) -rectifiable and let $\mu \in \mathcal{M}(A)$ be such that $\mu \not\ll \mathcal{H}_A^d$, then $\tilde{I}_d(\mu)$ exists and is infinite.*

Proof. Let $\mu \in \mathcal{M}(A)$ such that $\mu \not\ll \mathcal{H}_A^d$. Let $\mu = \mu^\perp + \mu^\ll$ be the Lebesgue decomposition of μ with respect to \mathcal{H}_A^d . Let K_1, \dots, K_N and $\varphi_1 : K_1 \rightarrow \mathbb{R}^p, \dots, \varphi_N : K_N \rightarrow \mathbb{R}^p$ be the compact subsets of \mathbb{R}^d and the corresponding maps with bi-Lipschitz constant less than 2 provided by the strong (\mathcal{H}^d, d) -rectifiability of A . Let $B = A \setminus \bigcup_{i=1}^N \varphi_i(K_i)$ and $s_0 = \dim B$. If $\mu(B) > 0$, then, by the equality of the capacity and Hausdorff dimensions (cf. [9]), $I_s(\mu) = \infty$ for all $s \in (s_0, d)$. Hence $\tilde{I}_d(\mu) = \infty$.

If $\mu(B) = 0$, then

$$0 < \mu^\perp(A) \leq \sum_{i=1}^N \mu^\perp(\varphi_i(K_i)).$$

Choose $j \in 1, \dots, N$ such that $\mu^\perp(\varphi_j(K_j)) > 0$, and define $\nu_j := \mu_{\varphi_j(K_j)}^\perp$. Since $\nu_j \perp \mathcal{H}_{\varphi_j(K_j)}^d$, it follows that $\varphi_{j\#}^{-1}\nu_j \perp \mathcal{H}^d$ and hence $\varphi_{j\#}^{-1}\nu_j \perp \mathcal{L}^d$. By Lemma 2.1 we have that $\tilde{I}_d(\varphi_{j\#}^{-1}\nu_j) = \infty$ and by (5) it follows that $\infty = \tilde{I}_d(\varphi_{j\#}\varphi_{j\#}^{-1}\nu_j) = \tilde{I}_d(\nu_j) \leq \tilde{I}_d(\mu)$. \square

Lemma 2.4. *Let $A \subset \mathbb{R}^p$ be strongly (\mathcal{H}^d, d) -rectifiable and $\mu \in \mathcal{M}(A)$ such that $\mu \ll \mathcal{H}_A^d$. Then*

$$\tilde{U}_d^\mu = 2^d d \frac{d\mu}{d\mathcal{H}_A^d} \quad \mathcal{H}_A^d\text{-a.e.}$$

Proof. As already noted, because A is strongly (\mathcal{H}^d, d) -rectifiable, it is (\mathcal{H}^d, d) -rectifiable. For any (\mathcal{H}^d, d) -rectifiable set, a density result (cf [9, ch. 16]) and the Radon-Nikodým Theorem give

$$\lim_{r \downarrow 0} \frac{\mathcal{H}_A^d(B(x, r))}{(2r)^d} = 1 \quad \text{and} \quad \lim_{r \downarrow 0} \frac{\mu(B(x, r))}{\mathcal{H}_A^d(B(x, r))} = \frac{d\mu}{d\mathcal{H}_A^d} \Big|_x < \infty \quad \text{for } \mathcal{H}_A^d\text{-a.e. } x.$$

For \mathcal{H}_A^d -a.e. x we then have $\sup_{r>0} \mu(B(x, r))/r^d < \infty$ and

$$D_\mu(x) = 2^d d \frac{d\mu}{d\mathcal{H}_A^d} \Big|_x.$$

Hence the order-two density exists and, by the result of Hinz in [7] mentioned earlier,

$$\tilde{U}_d^\mu(x) = 2^d d \frac{d\mu}{d\mathcal{H}_A^d} \Big|_x \quad \mathcal{H}_A^d\text{-a.e.}$$

\square

Lemma 2.5. *Let $A \subset \mathbb{R}^p$ be strongly (\mathcal{H}^d, d) -rectifiable and let $\mu \in \mathcal{M}(A)$ be such that $\mu \ll \mathcal{H}_A^d$ and $d\mu/d\mathcal{H}_A^d \notin L^2(\mathcal{H}_A^d)$, then $\tilde{I}_d(\mu)$ exists and is infinite.*

Proof. From Lemma 2.4 and Fatou's Lemma we immediately obtain

$$\begin{aligned} \infty &= 2^d d \int \left(\frac{d\mu}{d\mathcal{H}_A^d} \right)^2 d\mathcal{H}_A^d = \int \left(2^d d \frac{d\mu}{d\mathcal{H}_A^d} \right) d\mu = \int \tilde{U}_d^\mu d\mu \\ &= \int \left(\lim_{s \uparrow d} (d-s) \int \frac{1}{|x-y|^s} d\mu(y) \right) d\mu(x) \leq \liminf_{s \uparrow d} (d-s) \iint \frac{1}{|x-y|^s} d\mu(y) d\mu(x). \end{aligned}$$

\square

Lemma 2.6. *Let $A \subset \mathbb{R}^p$ be strongly (\mathcal{H}^d, d) -rectifiable. There is a constant C depending only on A such that for all $x \in \mathbb{R}^p$ and all $r > 0$*

$$\frac{\mathcal{H}_A^d(B(x, r))}{r^d} < C.$$

Proof. Let K_1, \dots, K_N and $\varphi_1 : K_1 \rightarrow \mathbb{R}^p, \dots, \varphi_N : K_N \rightarrow \mathbb{R}^p$ be the compact subsets of \mathbb{R}^d and the corresponding maps with bi-Lipschitz constant less than 2 provided by the strong (\mathcal{H}^d, d) -rectifiability of A . Since $\mathcal{H}^d(A) = \mathcal{H}^d(\bigcup_{i=1}^N \varphi_i(K_i))$ and since each φ_i is bijective, we have

$$\frac{\mathcal{H}_A^d(B(x, r))}{r^d} \leq \sum_{i=1}^N \frac{\mathcal{H}^d(\varphi_i(K_i) \cap B(x, r))}{r^d} = \sum_{i=1}^N \frac{\mathcal{H}^d(\varphi_i(K_i \cap \varphi_i^{-1}(B(x, r))))}{r^d} \leq \sum_{i=1}^N \frac{2^d \mathcal{H}^d(K_i \cap \varphi_i^{-1}(B(x, r)))}{r^d},$$

where the last inequality follows from (6). Since $\mathcal{H}^d(K_i \cap \varphi_i^{-1}(B(x, r))) \leq 2^{2d} r^d$, the claim holds with $C = 2^{3d} N$. \square

Lemma 2.7. *Let $A \subset \mathbb{R}^p$ be strongly (\mathcal{H}^d, d) -rectifiable and $\mu \in \mathcal{M}(A)$ be such that $\mu \ll \mathcal{H}_A^d$ and $d\mu/d\mathcal{H}_A^d \in L^2(\mathcal{H}_A^d)$, then $\tilde{I}_d(\mu)$ exists and*

$$\tilde{I}_d(\mu) = \int \tilde{U}_d^\mu d\mu.$$

Proof. The maximal function of μ with respect to \mathcal{H}_A^d may be expressed as

$$M_{\mathcal{H}_A^d} \mu(x) := \sup_{r>0} \frac{\mu(B(x, r))}{\mathcal{H}_A^d(B(x, r))} = \sup_{r>0} \frac{1}{\mathcal{H}_A^d(B(x, r))} \int_{B(x, r)} \frac{d\mu}{d\mathcal{H}_A^d} d\mathcal{H}_A^d.$$

The maximal function maps $L^2(\mathcal{H}_A^d)$ to itself and so $M_{\mathcal{H}_A^d} \mu(x) \in L^2(\mathcal{H}_A^d)$.

We construct a μ -integrable function that bounds $(d-s)U_s^\mu$ for all $s \in (0, d)$. Lemma 2.4 holds μ -a.e. and, for an x for which Lemma 2.4 holds, we follow an argument found in [9, ch. 2] to obtain

$$\begin{aligned} (d-s) \int \frac{1}{|x-y|^s} d\mu(y) &= (d-s) \int_0^\infty \mu\left(\left\{y \in \mathbb{R}^p : \frac{1}{|x-y|^s} > t\right\}\right) dt \\ &= (d-s)s \int_0^\infty \frac{\mu(B(x, r))}{r^{s+1}} dr \\ (7) \quad &= (d-s)s \int_0^{\text{diam } A} \frac{\mu(B(x, r))}{\mathcal{H}_A^d(B(x, r))} \frac{\mathcal{H}_A^d(B(x, r))}{r^d} r^{d-s-1} dr \\ (8) \quad &+ (d-s)s \int_{\text{diam } A}^\infty \frac{\mu(B(x, r))}{r^{s+1}} dr. \end{aligned}$$

The right hand side of (7) is bounded by $CM_{\mathcal{H}_A^d} \mu(x)s(\text{diam } A)^{d-s}$, where C is the constant established in Lemma 2.6. The quantity in (8) is bounded by $(d-s)\mu(\mathbb{R}^p)(\text{diam } A)^{-s}$. We may maximize these bounds over $s \in [0, d]$ to obtain a bound $(d-s)U_s^\mu$ of the form $C_1 M_{\mathcal{H}_A^d} \mu(x) + C_2 \mu(\mathbb{R}^p)$. The μ -integrability of this bound is established via the Cauchy-Schwarz inequality as follows

$$\int (C_1 M_{\mathcal{H}_A^d} \mu(x) + C_2 \mu(\mathbb{R}^p)) d\mu \leq C_1 \left\| M_{\mathcal{H}_A^d} \mu(x) \right\|_{2, \mathcal{H}_A^d} \left\| \frac{d\mu}{d\mathcal{H}_A^d} \right\|_{2, \mathcal{H}_A^d} + C_2 \mu(\mathbb{R}^p)^2 < \infty.$$

By dominated convergence the claim follows. \square

2.1. Proof of Theorem 1.2.

Proof of Theorem 1.2. Let A satisfy the hypotheses of Theorem 1.2. The first two claims of the theorem are proven in lemmas 2.3, 2.5 and 2.7.

Let ν denote the finite measure $(2^d d)^{-1} \mathcal{H}_A^d$. The set of measures with finite normalized d -energy are identified with the non-negative cone in $L^2(\nu)$ (denoted $L^2(\nu)_+$) via the map $\mu \leftrightarrow d\mu/d\nu$. Under this map we have $\tilde{I}_d(\mu) = \|d\mu/d\nu\|_{2, \nu}^2$. A measure μ of finite d -energy is a probability measure if and only if $\|d\mu/d\nu\|_{1, \nu} = 1$. The last claim in the theorem is proven by finding a unique, non-negative function f that minimizes $\|\cdot\|_{2, \nu}$ subject to the constraint $\|f\|_{1, \nu} = 1$. We address this problem using the following, standard Hilbert space argument.

The non-negative constant function $1/\nu(\mathbb{R}^p)$ satisfies the constraint $\|1/\nu(\mathbb{R}^p)\|_{1, \nu} = 1$. Let $f \in L^2(\nu)_+$ be such that $\|f\|_{1, \nu} = 1$ and $\|f\|_{2, \nu} \leq \|1/\nu(\mathbb{R}^p)\|_{2, \nu}$, then

$$\frac{1}{\nu(\mathbb{R}^p)} = \left\| \frac{f}{\nu(\mathbb{R}^p)} \right\|_{1, \nu} = \left\langle f, \frac{1}{\nu(\mathbb{R}^p)} \right\rangle_\nu \leq \|f\|_{2, \nu} \left\| \frac{1}{\nu(\mathbb{R}^p)} \right\|_{2, \nu} \leq \left\| \frac{1}{\nu(\mathbb{R}^p)} \right\|_{2, \nu}^2 = \frac{1}{\nu(\mathbb{R}^p)}.$$

Thus

$$\left\langle f, \frac{1}{\nu(\mathbb{R}^p)} \right\rangle_\nu = \|f\|_{2,\nu} \left\| \frac{1}{\nu(\mathbb{R}^p)} \right\|_{2,\nu}.$$

From the Cauchy-Schwarz inequality $f = 1/\nu(\mathbb{R}^p)$ ν -a.e. By the identification above, the measure, $\lambda^d := \mathcal{H}_A^d/\mathcal{H}^d(A) \in \mathcal{M}_1(A)$, uniquely minimizes \tilde{I}_d over $\mathcal{M}_1(A)$. \square

3. THE WEAK-STAR CONVERGENCE OF μ^s TO λ^d

Lemma 3.1. *Let $K \subset \mathbb{R}^d$ be a compact set. Then, for every $\eta > 0$, there is an $s_0 = s_0(\eta)$ such that, for any s and t satisfying $s_0 < s < t < d$ and any measure $\mu \in \mathcal{M}(K)$,*

$$(d-s)I_s(\mu) \leq (1+\eta) \left[(d-t)I_t(\mu) + \eta\mu(\mathbb{R}^d)^2 \right].$$

Proof. If $I_s(\mu) = \infty$, then $I_t(\mu) = \infty$ for $t > s$ and the lemma holds trivially. Now suppose that $I_s(\mu) < \infty$ for some s such that $(d-t)c(t, d) > \omega_d/2$ for all $t \in (s, d)$ and observe that

$$\begin{aligned} (d-s)I_s(\mu) &= (d-s)c(s, d) \int_{\mathbb{R}^d} |\xi|^{s-d} |\hat{\mu}(\xi)|^2 d\mathcal{L}^d(\xi) \\ (9) \qquad &= \frac{(d-s)c(s, d)}{(d-t)c(t, d)} (d-t)c(t, d) \int_{\mathbb{R}^d} |\xi|^{s-d} |\hat{\mu}(\xi)|^2 d\mathcal{L}^d(\xi). \end{aligned}$$

We may approximate the integral in (9) as follows.

$$\begin{aligned} &\int_{\mathbb{R}^d} |\xi|^{s-d} |\hat{\mu}(\xi)|^2 d\mathcal{L}^d(\xi) \\ &= \int_{|\xi| \leq 1} |\xi|^{s-d} |\hat{\mu}(\xi)|^2 d\mathcal{L}^d(\xi) + \int_{|\xi| > 1} |\xi|^{s-d} |\hat{\mu}(\xi)|^2 d\mathcal{L}^d(\xi) \\ &\leq \int_{|\xi| \leq 1} (|\xi|^{s-d} - |\xi|^{t-d}) |\hat{\mu}(\xi)|^2 d\mathcal{L}^d(\xi) + \int_{|\xi| \leq 1} |\xi|^{t-d} |\hat{\mu}(\xi)|^2 d\mathcal{L}^d(\xi) + \int_{|\xi| > 1} |\xi|^{t-d} |\hat{\mu}(\xi)|^2 d\mathcal{L}^d(\xi) \\ &\leq \mu(\mathbb{R}^d)^2 \int_{|\xi| \leq 1} (|\xi|^{s-d} - |\xi|^{t-d}) d\mathcal{L}^d(\xi) + \int_{\mathbb{R}^d} |\xi|^{t-d} |\hat{\mu}(\xi)|^2 d\mathcal{L}^d(\xi). \end{aligned}$$

By (4) we may pick $s_0 \in (0, d)$ high enough so that, for any s and t satisfying $s_0 < s < t < d$

$$\frac{(d-s)c(s, d)}{(d-t)c(t, d)} < 1 + \eta, \quad (d-t)c(t, d) < 2\omega_d,$$

and

$$\left| \int_{|\xi| \leq 1} (|\xi|^{s-d} - |\xi|^{t-d}) d\mathcal{L}^d(\xi) \right| < \frac{\eta}{2\omega_d}.$$

\square

The following generalization of Lemma 3.1 will be applied repeatedly to measures supported on the bi-Lipschitz image of a compact set, $K \subset \mathbb{R}^d$. Let $\mu \in \mathcal{M}(\varphi(K))$ be such a measure. Using (5) to bound the s -energy of $\varphi_\#^{-1}\mu$, applying Lemma 3.1 to $\varphi_\#^{-1}\mu$, and then using (5) again to bound the t -energy of the measure $\varphi_\#\varphi_\#^{-1}\mu = \mu$ we obtain the following.

Corollary 3.2. *Let $K \subset \mathbb{R}^d$ be a compact set and suppose $\varphi : K \rightarrow \mathbb{R}^p$ is bi-Lipschitz with constant L . Then, for every $\eta > 0$ there is an $s_0 = s_0(\eta)$ such that for any s and t satisfying $s_0 < s < t < d$ and any measure $\mu \in \mathcal{M}(\varphi(K))$, we have*

$$(d-s)I_s(\mu) \leq L^d(1+\eta) \left[L^d(d-t)I_t(\mu) + \eta\mu(\mathbb{R}^p)^2 \right].$$

In the proof of the following proposition we shall use the Principle of Descent (cf. [8, ch.1 §4]), a consequence of which is that, if $s \in (0, d)$ and if a sequence of compactly supported Radon measures $\{\mu_n\}_{n=1}^\infty$ converges in the weak-star topology to ψ , then $I_s(\psi) \leq \liminf_{n \rightarrow \infty} I_s(\mu_n)$.

Proposition 3.3 is a simple case of Theorem 1.3 and its proof illustrates the approach used in the proof of Theorem 1.3.

Proposition 3.3. *Let $A \subset \mathbb{R}^d$ be a compact set such that $\mathcal{H}^d(A) > 0$. Let μ^s denote the s -equilibrium measure supported on A . Then $\mu^s \xrightarrow{*} \lambda^d := \mathcal{H}_A^d / \mathcal{H}^d(A)$ as $s \uparrow d$.*

Proof. Let $\psi \in \mathcal{M}_1(A)$ be a weak-star cluster point of μ^s as $s \uparrow d$. Let $\{s_n\}_{n=1}^\infty \uparrow d$ such that $\mu^{s_n} \xrightarrow{*} \psi$ as $n \rightarrow \infty$. Let $\eta > 0$ be arbitrary, s_0 be as provided by Lemma 3.1, and let $s \in (s_0, d)$. We have

$$\begin{aligned} (d-s)I_s(\psi) &\leq \liminf_{n \rightarrow \infty} (d-s)I_s(\mu^{s_n}) \\ &\leq \liminf_{n \rightarrow \infty} (1+\eta) [(d-s_n)I_{s_n}(\mu^{s_n}) + \eta] \\ &\leq \liminf_{n \rightarrow \infty} (1+\eta) [(d-s_n)I_{s_n}(\lambda^d) + \eta] \\ &= (1+\eta) [\tilde{I}_d(\lambda^d) + \eta], \end{aligned}$$

where the first inequality is an application of the Principle of Descent. The second inequality follows from Lemma 3.1 where t in the statement of the lemma is chosen to be s_n , and the third from the minimality of $I_{s_n}(\mu^{s_n})$.

The variable s may be taken arbitrarily close to d , and so $\tilde{I}_d(\psi) \leq (1+\eta)[\tilde{I}_d(\lambda^d) + \eta]$. The variable η was also chosen arbitrarily and we conclude $\tilde{I}_d(\psi) \leq \tilde{I}_d(\lambda^d)$. Theorem (1.2) ensures that λ^d is the unique probability measure that minimizes \tilde{I}_d , and so $\psi = \lambda^d$. Since this holds for any weak-star cluster point, the proposition is proven. \square

The rest of the paper shall employ several classical results from potential theory (cf. [8]). Let \mathcal{E}_s denote the set of all signed Radon measures supported in \mathbb{R}^p of finite total variation such that μ is an element of \mathcal{E}_s if and only if $I_s(|\mu|) < \infty$. The set \mathcal{E}_s is a vector space, and, when combined with the following bilinear form

$$I_s(\mu, \nu) = \iint \frac{1}{|x-y|^s} d\mu(x) d\nu(y),$$

is a pre-Hilbert space. Further, for $\mu \in \mathcal{E}_s$

$$I_s(\mu) = \int U_s^\mu d\mu.$$

A property is said to hold *approximately everywhere*, if it holds everywhere except on a set of points contained in a compact set that supports no non-trivial measures in \mathcal{E}_s . For $s < \dim A$, the equilibrium measure μ^s satisfies $U_s^{\mu^s} = I_s(\mu^s)$ approximately everywhere in $\text{supp}\{\mu^s\}$. In particular $U_s^{\mu^s} = I_s(\mu^s)$ μ^s -a.e.

The proof of Theorem 1.3 follows essentially the same approach used in the proof of Proposition 3.3. The only technical hurdle is to establish an analog of Lemma 3.1 for the case when A is strongly (\mathcal{H}^d, d) -rectifiable and of lower dimension than that of the embedding space, \mathbb{R}^p . This is accomplished by breaking A into near isometries of compact subsets of \mathbb{R}^d , establishing the desired estimate one each piece, and showing that the pieces can be glued back together without affecting the estimate. This is the content of lemmas 3.4, 3.5 and 3.6.

Lemma 3.4. *Let $A \subset \mathbb{R}^p$ be a compact, strongly (\mathcal{H}^d, d) -rectifiable set such that $\mathcal{H}^d(A) > 0$. Let $K \subset \mathbb{R}^d$ be compact, and $\varphi : K \rightarrow \mathbb{R}^p$ a bi-Lipschitz map such that $\varphi(K) \subset A$. Then, for every $\varepsilon > 0$, there is an $s_0 = s_0(\varepsilon)$ and a constant $C_{K,\varphi} = C_{K,\varphi}(A, K, \varphi)$ such that, for any Borel set $B \subset \mathbb{R}^p$ satisfying $\mathcal{H}_A^d(\partial B) = 0$ and any $s \in (s_0, d)$,*

$$\limsup_{t \uparrow d} (d-s)I_s(\mu_{B \cap \varphi(K)}^t) \leq C_{K,\varphi} \sqrt{\mathcal{H}_A^d(B)} + \varepsilon.$$

The boundary, ∂B , is computed in the usual topology on \mathbb{R}^p .

Proof. Without loss of generality assume $\varepsilon \in (0, 1)$. Let $B \subset \mathbb{R}^p$ be a Borel set such that $\mathcal{H}_A^d(\partial B) = 0$. Observe that

$$(10) \quad I_t(\mu_{B \cap \varphi(K)}^t) = \int_{B \cap \varphi(K)} U_t^{\mu_{B \cap \varphi(K)}^t} d\mu^t \leq \int_{B \cap \varphi(K)} U_t^{\mu^t} d\mu^t = I_t(\mu^t) \mu^t(B \cap \varphi(K)).$$

We bound the quantity $\limsup_{t \uparrow d} \mu^t(B \cap \varphi(K))$ as follows. Let $\psi \in \mathcal{M}(A)$ be a weak-star cluster point of $\mu_{B \cap \varphi(K)}^t$ as $t \uparrow d$, and let $\{t_n\}_{n=1}^\infty \uparrow d$ such that $\mu_{B \cap \varphi(K)}^{t_n} \xrightarrow{*} \psi$ as $n \rightarrow \infty$. Let L denote the bi-Lipschitz constant of

φ . Choose \tilde{s}_0 so that Corollary 3.2 applied to Radon measures with supported in $\varphi(K)$ holds for $\eta = 1$. Let $\lambda^d := \mathcal{H}_A^d / \mathcal{H}^d(A)$ denote the minimizer of \tilde{I}_d over $\mathcal{M}_1(A)$. For any $s \in (\tilde{s}_0, d)$,

$$\begin{aligned} (d-s)I_s(\psi) &\leq \liminf_{n \rightarrow \infty} (d-s)I_s(\mu_{B \cap \varphi(K)}^{t_n}) \\ &\leq \liminf_{n \rightarrow \infty} 2L^d \left[(d-t_n)L^d I_{t_n}(\mu^{t_n}) + 1 \right] \\ &\leq \liminf_{n \rightarrow \infty} 2L^d \left[(d-t_n)L^d I_{t_n}(\lambda^d) + 1 \right] \\ &= 2L^{2d} \tilde{I}_d(\lambda^d) + 2L^d =: M < \infty. \end{aligned}$$

The first inequality follows from the Principle of Descent, the second from Corollary 3.2 and the inequality, $I_s(\mu_{B \cap \varphi(K)}^{t_n}) \leq I_s(\mu^{t_n})$, and the third from the minimality of $I_{t_n}(\mu^{t_n})$. Letting $s \uparrow d$ we see that, for any weak-star cluster point ψ of $\mu_{B \cap \varphi(K)}^{t_n}$ (as $t \uparrow d$), $\tilde{I}_d(\psi) \leq M$. Theorem 1.2 ensures that $\psi \ll \mathcal{H}_A^d$, and so $\psi(\partial B) = 0$, implying $\mu^{t_n}(B \cap \varphi(K)) = \mu_{B \cap \varphi(K)}^{t_n}(B) \rightarrow \psi(B)$ as $n \rightarrow \infty$.

The set $\bar{B} \cap A$ is strongly d -rectifiable, and if $\psi(B) > 0$, then $\mathcal{H}_A^d(B) > 0$, implying $\mathcal{H}^d(\bar{B} \cap A) > 0$ and by Theorem 1.2, \tilde{I}_d is minimized over $\mathcal{M}_1(\bar{B} \cap A)$ by $\lambda^{d, \bar{B} \cap A} := \mathcal{H}_{\bar{B} \cap A}^d / \mathcal{H}^d(\bar{B} \cap A)$. We then have

$$\frac{2^d d}{\mathcal{H}_A^d(B)} = \frac{2^d d}{\mathcal{H}_A^d(\bar{B})} = \frac{2^d d}{\mathcal{H}^d(\bar{B} \cap A)} = \tilde{I}_d(\lambda^{d, \bar{B} \cap A}) \leq \tilde{I}_d\left(\frac{\psi}{\psi(B)}\right) = \tilde{I}_d\left(\frac{\psi}{\psi(B)}\right) \leq \frac{M}{\psi(B)^2},$$

and we may conclude

$$\psi(B) \leq \sqrt{\frac{M}{2^d d} \mathcal{H}_A^d(B)}.$$

(If $\psi(B) = 0$, then the above inequality holds trivially.) It follows from the above inequality and (10) that for any Borel set $B \subset \mathbb{R}^p$ with $\mathcal{H}_A^d(\partial B) = 0$ we have

$$(11) \quad \limsup_{t \uparrow d} (d-t)I_t(\mu_{B \cap \varphi(K)}^t) \leq \limsup_{t \uparrow d} (d-t)I_t(\mu^t) \limsup_{t \uparrow d} \mu^t(B \cap \varphi(K)) \leq \tilde{I}_d(\lambda^d) \sqrt{\frac{M}{2^d d}} \sqrt{\mathcal{H}_A^d(B)}.$$

We complete the proof of this lemma by appealing to Corollary 3.2 applied to measures supported on $\varphi(K)$ with $\eta = \varepsilon/2L^d$. If s_0 is chosen so that Corollary 3.2 holds, then, for any $s \in (s_0, d)$ and $t \in (s, d)$,

$$\begin{aligned} (d-s)I_s(\mu_{B \cap \varphi(K)}^t) &\leq L^d \left[\left(1 + \frac{\varepsilon}{2L^d}\right) \left(L^d(d-t)I_t(\mu_{B \cap \varphi(K)}^t) + \frac{\varepsilon}{2L^d}\right) \right] \\ &\leq 2L^{2d}(d-t)I_t(\mu_{B \cap \varphi(K)}^t) + \varepsilon. \end{aligned}$$

Taking the limit superior of both sides as $t \uparrow d$ and appealing to (11) completes the proof with $C_{K, \varphi} = 2L^{2d} \tilde{I}_d(\lambda^d) \sqrt{M/2^d d}$. \square

Lemma 3.5. *Let $A \subset \mathbb{R}^p$ be a compact, strongly (\mathcal{H}^d, d) -rectifiable set such that $\mathcal{H}^d(A) > 0$. Then, for every $\varepsilon > 0$, there exists a finite collection of compact subsets of \mathbb{R}^d $\tilde{K}_1, \dots, \tilde{K}_N$ and a corresponding set of bi-Lipschitz maps $\tilde{\varphi}_1 : \tilde{K}_1 \rightarrow \mathbb{R}^p, \dots, \tilde{\varphi}_N : \tilde{K}_N \rightarrow \mathbb{R}^p$ each with bi-Lipschitz constant less than $1 + \varepsilon$, such that*

1. $\tilde{\varphi}_i(\tilde{K}_i) \cap \tilde{\varphi}_j(\tilde{K}_j) = \emptyset$ for $i \neq j$, and
2. there is an $s_0 = s_0(\varepsilon) \in (0, d)$, such that for $\tilde{B} := A \setminus \bigcup_{i=1}^N \tilde{\varphi}_i(\tilde{K}_i)$ and all $s \in (s_0, d)$ we have

$$\limsup_{t \uparrow d} (d-s)I_s(\mu_B^t) \leq \frac{\varepsilon}{N}.$$

Proof. Without loss of generality assume $\varepsilon \in (0, 1)$. Since A is strongly (\mathcal{H}^d, d) -rectifiable, we may find a set, $A_0 \subset \mathbb{R}^p$, compact sets $K_1, \dots, K_N \subset \mathbb{R}^d$ and bi-Lipschitz maps $\varphi_1 : K_1 \rightarrow \mathbb{R}^p, \dots, \varphi_N : K_N \rightarrow \mathbb{R}^p$ with constant less than $1 + \varepsilon$ such that $A = \bigcup_{i=1}^N \varphi_i(K_i) \cup A_0$, where $\dim A_0 < d$, and $\mathcal{H}^d(\varphi_i(K_i) \cap \varphi_j(K_j)) = 0$. Let $\delta = \varepsilon^2/4N^2 \in (0, 1)$. The set $E = \bigcup_{i \neq j} (\varphi_i(K_i) \cap \varphi_j(K_j))$ is a compact set of \mathcal{H}_A^d -measure 0. Since \mathcal{H}_A^d is Radon, there is an open set O such that $E \subset O$ and $\mathcal{H}_A^d(O) < \delta N^{-4} \left(\max \{C_{K_1, \varphi_1}, \dots, C_{K_N, \varphi_N}\} \right)^{-2}$ where C_{K_i, φ_i} is the constant provided by Lemma 3.4 applied to $\varphi_i(K_i) \subset A$.

For any point $x \in E$, we may find a non-empty open ball $B(x, R)^0 \subset O$. Since $\partial B(x, r_1) \cap \partial B(x, r_2) = \emptyset$ for any $r_1 \neq r_2$ and since \mathcal{H}_A^d is a finite measure, all but a countable set of values of $r \in (0, R)$ must be such that $\mathcal{H}_A^d(\partial B(x, r)) = 0$. Construct an open cover of E as follows.

$$\Omega = \left\{ B(x, r)^0 : x \in E, B(x, r)^0 \subset O, \mathcal{H}_A^d(\partial B(x, r)) = 0 \right\}.$$

Choose a finite sub-cover $\Omega' \subset \Omega$, of E . Let $B = \bigcup_{b \in \Omega'} b$. Since $\partial B \subset \bigcup_{b \in \Omega'} \partial b$, we have that $\mathcal{H}_A^d(\partial B) = 0$. Let $B_i = B \cap \varphi_i(K_i)$. For any $s, t \in (0, d)$ with $t > \max\{s, \dim A_0\}$ we have, by the equality of the Hausdorff and capacitory dimensions, that $\mu^t(A_0) = 0$ and hence

$$(d-s)I_s(\mu_B^t) \leq (d-s)I_s\left(\mu_{A_0}^t + \sum_{i=1}^N \mu_{B_i}^t\right) = \sum_{i,j=1}^N (d-s)I_s(\mu_{B_i}^t, \mu_{B_j}^t).$$

By Jensen's inequality followed by the Cauchy-Schwarz inequality applied to the inner-product $I_s(\cdot, \cdot)$ we have

$$\left[\frac{1}{N^2} \sum_{i,j=1}^N (d-s)I_s(\mu_{B_i}^t, \mu_{B_j}^t) \right]^2 \leq \frac{1}{N^2} \sum_{i,j=1}^N [(d-s)I_s(\mu_{B_i}^t, \mu_{B_j}^t)]^2 \leq \frac{1}{N^2} \sum_{i,j=1}^N (d-s)I_s(\mu_{B_i}^t)(d-s)I_s(\mu_{B_j}^t).$$

Let $s_0 = \max\{\dim A_0, s_{0,1}, \dots, s_{0,N}\}$, where $s_{0,i}$ is the value of s_0 provided by Lemma 3.4 applied to $\varphi_i(K_i) \subset A$, and where the value of ε in the statement of Lemma 3.4 is chosen to be δ/N^2 . Combining the previous bounds gives, for $s \in (s_0, d)$,

$$\begin{aligned} \left[\limsup_{t \uparrow d} (d-s)I_s(\mu_B^t) \right]^2 &\leq N^2 \sum_{i,j=1}^N \limsup_{t \uparrow d} (d-s)I_s(\mu_{B_i}^t) \limsup_{t \uparrow d} (d-s)I_s(\mu_{B_j}^t) \\ &\leq N^2 \sum_{i,j=1}^N \left(C_{K_i, \varphi_i} \sqrt{\frac{\delta}{N^4 (C_{K_i, \varphi_i})^2} + \frac{\delta}{N^2}} + \frac{\delta}{N^2} \right) \left(C_{K_j, \varphi_j} \sqrt{\frac{\delta}{N^4 (C_{K_j, \varphi_j})^2} + \frac{\delta}{N^2}} + \frac{\delta}{N^2} \right) \\ &= N^2 \sum_{i,j=1}^N \left(\frac{\sqrt{\delta} + \delta}{N^2} \right)^2 \\ &\leq 4\delta = \left(\frac{\varepsilon}{N} \right)^2. \end{aligned}$$

The value of s_0 , the set $\tilde{B} := (B \cap A) \cup A_0$, the compact sets $\tilde{K}_i := K_i \setminus \varphi_i^{-1}(B)$, and the bi-Lipschitz maps $\tilde{\varphi}_i := \varphi_i|_{\tilde{K}_i}$ satisfy the properties claimed in the lemma for the value of ε given. \square

Lemma 3.6. *Let $A \subset \mathbb{R}^p$ be a strongly (\mathcal{H}^d, d) -rectifiable, compact set such that $\mathcal{H}^d(A) > 0$. Then, for every $\eta > 0$, there is an $s_0 = s_0(\eta)$, such that for all $s \in (s_0, d)$ we have*

$$\limsup_{t \uparrow d} (d-s)I_s(\mu^t) \leq (1+\eta) \limsup_{t \uparrow d} (d-t)I_t(\mu^t) + \eta.$$

Proof. Let $\lambda^d := \mathcal{H}_A^d / \mathcal{H}^d(A)$ denote the unique minimizer of \tilde{I}_d over $\mathcal{M}_1(A)$. Let $\eta > 0$. Choose $\varepsilon \in (0, 1)$ such that

$$(12) \quad \max \left\{ \left(\varepsilon \left[2 + (1+\varepsilon)^{d+1} \right] + 2 \sqrt{\varepsilon(1+\varepsilon)^{2d+1} \tilde{I}_d(\lambda^d) + \varepsilon^2(1+\varepsilon)^{d+1}} \right), \left((1+\varepsilon)^{2d+1} - 1 \right) \right\} < \eta.$$

From Lemma 3.5 there is an $s_1 \in (0, d)$, a sequence of compact sets $\tilde{K}_1, \dots, \tilde{K}_N \subset \mathbb{R}^d$ and a sequence of bi-Lipschitz maps $\tilde{\varphi}_1 : \tilde{K}_1 \rightarrow \mathbb{R}^p, \dots, \tilde{\varphi}_N : \tilde{K}_N \rightarrow \mathbb{R}^p$ each with constant less than $1+\varepsilon$ such that $\tilde{\varphi}_i(\tilde{K}_i) \cap \tilde{\varphi}_j(\tilde{K}_j) = \emptyset$ for $i \neq j$, and $\tilde{B} := A \setminus \bigcup_{i=1}^N \tilde{\varphi}_i(\tilde{K}_i)$ satisfies the following for all $s \in (s_1, d)$

$$\limsup_{t \uparrow d} (d-s)I_s(\mu_{\tilde{B}}^t) \leq \frac{\varepsilon}{N}.$$

For $s \in (s_1, d)$ we have

$$(13) \quad \limsup_{t \uparrow d} (d-s)I_s(\mu^t) = \limsup_{t \uparrow d} (d-s)I_s\left(\mu_B^t + \sum_{i=1}^N \mu_{\tilde{\varphi}_i(\tilde{K}_i)}^t\right) \\ \leq \limsup_{t \uparrow d} (d-s)I_s(\mu_B^t)$$

$$(14) \quad + 2 \limsup_{t \uparrow d} \sum_{i=1}^N (d-s)I_s(\mu_B^t, \mu_{\tilde{\varphi}_i(\tilde{K}_i)}^t)$$

$$(15) \quad + \limsup_{t \uparrow d} \sum_{\substack{i,j=1 \\ i \neq j}}^N (d-s)I_s(\mu_{\tilde{\varphi}_i(\tilde{K}_i)}^t, \mu_{\tilde{\varphi}_j(\tilde{K}_j)}^t)$$

$$(16) \quad + \limsup_{t \uparrow d} \sum_{i=1}^N (d-s)I_s(\mu_{\tilde{\varphi}_i(\tilde{K}_i)}^t).$$

We next find upper bounds for each of the terms in (13–16). First, Lemma 3.5 implies that, for $s \in (s_1, d)$, expression (13) is less than ε/N .

Second, using Jensen's inequality and the Cauchy-Schwarz inequality in the same manner as in the proof of Lemma 3.5 we have

$$\sum_{i=1}^N (d-s)I_s(\mu_B^t, \mu_{\tilde{\varphi}_i(\tilde{K}_i)}^t) \leq \sqrt{N(d-s)I_s(\mu_B^t) \sum_{i=1}^N (d-s)I_s(\mu_{\tilde{\varphi}_i(\tilde{K}_i)}^t)}.$$

Since each $\tilde{\varphi}_i$ is bi-Lipschitz with constant $(1 + \varepsilon)$, Corollary 3.2 (with the values of η and L as stated in the corollary chosen to be ε and $1 + \varepsilon$ respectively) ensures that there is some $s_2 \in (s_1, d)$ such that, for $s_2 < s < t < d$, we have

$$(17) \quad (d-s)I_s(\mu_{\tilde{\varphi}_i(\tilde{K}_i)}^t) \leq (1 + \varepsilon)^{2d+1} (d-t)I_t(\mu_{\tilde{\varphi}_i(\tilde{K}_i)}^t) + \varepsilon(1 + \varepsilon)^{d+1} \mu_{\tilde{\varphi}_i(\tilde{K}_i)}^t(\mathbb{R}^p)^2.$$

Then (17), together with the bound for (13), implies that expression (14) is bounded above by

$$(18) \quad 2 \sqrt{N \frac{\varepsilon}{N} \limsup_{t \uparrow d} \left[(1 + \varepsilon)^{2d+1} \sum_{i=1}^N (d-t)I_t(\mu_{\tilde{\varphi}_i(\tilde{K}_i)}^t) + \varepsilon(1 + \varepsilon)^{d+1} \sum_{i=1}^N \mu_{\tilde{\varphi}_i(\tilde{K}_i)}^t(\mathbb{R}^p)^2 \right]}$$

Using

$$\limsup_{t \uparrow d} \sum_{i=1}^N (d-t)I_t(\mu_{\tilde{\varphi}_i(\tilde{K}_i)}^t) \leq \limsup_{t \uparrow d} (d-t)I_t(\mu^t) \leq \limsup_{t \uparrow d} (d-t)I_t(\lambda^d) = \tilde{I}_d(\lambda^d)$$

it follows that, for $s \in (s_2, d)$, expression (14) is bounded above by

$$2 \sqrt{\varepsilon \left[(1 + \varepsilon)^{2d+1} \tilde{I}_d(\lambda^d) + \varepsilon(1 + \varepsilon)^{d+1} \right]}.$$

We bound (15) as follows. For $1 \leq i \neq j \leq N$, let $D_{i,j} = \text{dist}(\tilde{\varphi}_i(\tilde{K}_i), \tilde{\varphi}_j(\tilde{K}_j)) > 0$ and let $s_{i,j} \in (0, d)$ be such that $(d-s)D_{i,j}^{-s} \leq \varepsilon/N^2$ for all $s \in (s_{i,j}, d)$. For such an s , $(d-s)I_s(\nu_1, \nu_2) \leq \nu_1(\mathbb{R}^p)\nu_2(\mathbb{R}^p)\varepsilon/N^2$, for any $\nu_1, \nu_2 \in \mathcal{M}(A)$ supported on $\tilde{\varphi}_i(\tilde{K}_i)$ and $\tilde{\varphi}_j(\tilde{K}_j)$ respectively. Let $s_0 := \max\{s_2, s_{i,j} : i \neq j\}$. For all $s \in (s_0, d)$,

$$\sum_{\substack{i,j=1 \\ i \neq j}}^N (d-s)I_s(\mu_{\tilde{\varphi}_i(\tilde{K}_i)}^t, \mu_{\tilde{\varphi}_j(\tilde{K}_j)}^t) < \varepsilon.$$

From (17) we have the following bound for (16)

$$\begin{aligned} \sum_{i=1}^N (d-s)I_s(\mu_{\tilde{\varphi}_i(\tilde{K}_i)}^t) &\leq (1 + \varepsilon)^{2d+1} \left(\sum_{i=1}^N (d-t)I_t(\mu_{\tilde{\varphi}_i(\tilde{K}_i)}^t) \right) + \varepsilon(1 + \varepsilon)^{d+1} \left(\sum_{i=1}^N \mu_{\tilde{\varphi}_i(\tilde{K}_i)}^t(\mathbb{R}^p)^2 \right) \\ &\leq (1 + \varepsilon)^{2d+1} (d-t)I_t(\mu^t) + \varepsilon(1 + \varepsilon)^{d+1}. \end{aligned}$$

For $s \in (s_0, d)$, the preceding estimates, together with (12), gives

$$\begin{aligned} \limsup_{t \uparrow d} (d-s)I_s(\mu^t) &\leq \left[\varepsilon \left[2 + (1+\varepsilon)^{d+1} \right] + 2 \sqrt{\varepsilon(1+\varepsilon)^{2d+1} \tilde{I}_d(\lambda^d) + \varepsilon^2(1+\varepsilon)^{d+1}} \right] \\ &\quad + \left[(1+\varepsilon)^{2d+1} \right] \limsup_{t \uparrow d} (d-t)I_t(\mu^t) \\ &\leq \eta + (1+\eta) \limsup_{t \uparrow d} (d-t)I_t(\mu^t). \end{aligned}$$

□

3.1. Proof of Theorem 1.3.

proof of theorem 1.3. Let A satisfy the hypotheses of Theorem 1.3 and hence of Theorem 1.2. Let $\lambda^d := \mathcal{H}_A^d / \mathcal{H}^d(A)$ denote the unique minimizer of \tilde{I}_d over $\mathcal{M}_1(A)$. Let ψ be any weak-star cluster point of μ^s as $s \uparrow d$, and let $\{s_n\}_{n=1}^\infty \uparrow d$ such that $\mu^{s_n} \xrightarrow{*} \psi$. Let $\eta > 0$ be arbitrary. Let s_0 be the value provided by lemma 3.6 for this choice of η . For any $s \in (s_0, d)$, we have

$$\begin{aligned} (d-s)I_s(\psi) &\leq \liminf_{n \rightarrow \infty} (d-s)I_s(\mu^{s_n}) \\ &\leq \limsup_{n \rightarrow \infty} (d-s_n)I_{s_n}(\mu^{s_n})(1+\eta) + \eta \\ &\leq \limsup_{n \rightarrow \infty} (d-s_n)I_{s_n}(\lambda^d)(1+\eta) + \eta \\ &= (1+\eta)\tilde{I}_d(\lambda^d) + \eta. \end{aligned}$$

As in the proof of Proposition 3.3, the first inequality follows from the Principle of Descent, the second from Lemma 3.6, and the third from the minimality of $I_{s_n}(\mu^{s_n})$. Since s may be chosen arbitrarily close to d , $\tilde{I}_d(\psi) \leq (1+\eta)\tilde{I}_d(\lambda^d) + \eta$. Since η was also arbitrarily chosen, $\tilde{I}_d(\psi) \leq \tilde{I}_d(\lambda^d)$. The uniqueness of the minimizer λ^d ensured by Theorem 1.2 proves that $\psi = \lambda^d$ and is sufficient to prove Theorem 1.3. □

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